Contents

Prerequisites

This document assumes the reader has background knowledge and strong foundations in the following:

- **Introductory algebra**
  - Solving linear and quadratic equations
  - Working with radicals and exponents
  - Function notation and properties
- **Basic trigonometry**
  - $\sin, \cos, \tan$ functions and their inverses
  - Radians and degree conversion
  - Standard position and reference angles
  - Pythagorean Theorem and trigonometric identities
  - Unit circle and special angles $(0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2})$
- **The imaginary unit** i
  - Definition: $i = \sqrt{-1}$ and $i^2 = -1$
  - Powers of i: i, i^2, i^3, i^4 and their cyclic nature

The following topics are not essential at all, but would help the reader better understand the document:

- **Polar coordinates (basic)**
  - Converting between Cartesian (x, y) and polar (r, θ) coordinates
  - Understanding r as distance/radius and θ as angle
  - Plotting points in polar form
  - Negative radii and angle adjustments

Notations used throughout the packet:

- i: The solution to the equation $x^2 = -1$
- $a \in b$: a Is in the set b
- $a \forall b$: Statement a holds for all values of b
- $\mathbb{R}$: The set of real numbers
- $\mathbb{Z}$: The set of integers
- $\mathbb{C}$: The set of complex numbers

1 Introduction to Complex Numbers and the Complex Plane

Note 1.1 - Definition of a Complex Number

A **complex number** is defined as a number that can be expressed as $a + bi$, where a and b are real numbers, and i is the principal square root of -1. The **complex set** of numbers is denoted by the symbol $\mathbb{C}$. In other words,

$$z \in \mathbb{C} \text{ if and only if } z = a + bi, \text{ where } a, b \in \mathbb{R}.$$

Since a and b can equal 0, the complex set includes the set of real numbers and the set of purely imaginary numbers as well.

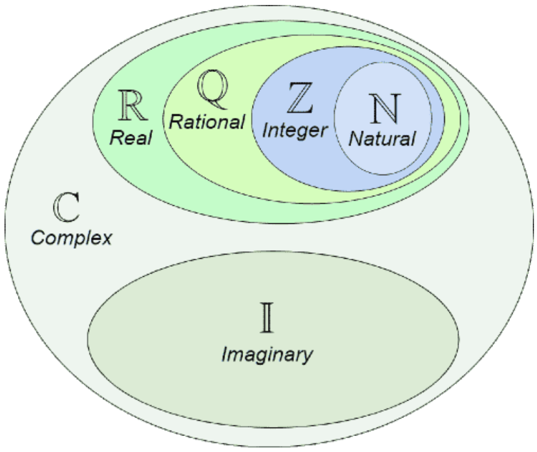


Figure 1.1: Hierarchy of Sets of Numbers

For example, $1 \in \mathbb{C}$ since it can be expressed as $1 + 0i$. $\pi i \in \mathbb{C}$ since it can be expressed as $0 + \pi i$.
If $z \in \mathbb{C}$ and $z = a + bi$:

- The Real Part of z is denoted by $Re(z)$. Here, $Re(z) = a$.
- The Imaginary Part of z is denoted by $Im(z)$. Here, $Im(z) = b$. Notice how $Im(z)$ does not include i.

Example 1.1 - Real and Imaginary Parts of a Complex Number

Suppose $w = 3 - i\sqrt{7}$.
(a) What is $Re(w)$?
(b) What is $Im(w)$?

Solution to (a) : The real part of w is $\boxed{3}$.

Solution to (b) : The imaginary part of w is $\boxed{-\sqrt{7}}$.

The real and imaginary parts are important in operating with complex numbers. When adding/subtracting multiple complex numbers, we combine their real parts and imaginary parts.

Note 1.2 - Adding and Subtracting Complex numbers

Let $z = a + bi$ and $w = c + di$. Then,

- $z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$
- $z - w = (a + bi) - (c + di) = (a - c) + (b - d)i$

For example, $(2 + 3i) - (4 - 5i) = (2 - 4) + (3 - (-5))i = -2 + 8i$

Adding and subtracting complex numbers is quite simple, as all you have to do is to combine the different parts together. This process is similar to combining like terms when simplifying expressions, such as $3 + x + y - 2x - 7y = (3) + (1 - 2)x + (1 - 7)y = 3 - x - 6y$.

Multiplying complex numbers requires a bit more effort but can be done using the so-called "FOIL" method, illustrated by the figure below:

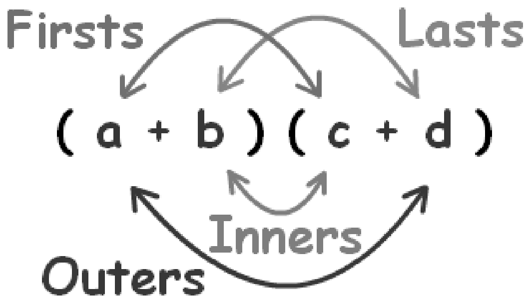


Figure 1.2: "FOIL" method

Note 1.3 - Multiplying Complex Numbers

Let $z = a + bi$ and $w = c + di$. Then,

- $z \cdot w = (a + bi)(c + di) = a(c) + a(di) + bi(c) + bi(di) = ac + adi + bci + bdi^2 = ac + (ad + bc)i - bd$, since $i^2 = -1$. We can see that $Re(z \cdot w) = ac - bd$, and $Im(z \cdot w) = ad + bc$.

This is more easily expressed with an example. Suppose we want to multiply $2 + 5i$ by $3 - 4i$.

$$(2 + 5i)(3 - 4i) = 6 - 8i + 15i - 20i^2 = 6 + 7i + 20 = 26 + 7i.$$

Finally, we must discuss dividing with complex numbers. This process requires careful manipulation using the **complex conjugate**. If we let $z = a+bi$, then its complex conjugate is $\overline{z} = a - bi$. The only difference is that the sign of the imaginary part is reversed. These are called **conjugate pairs**. But why is this useful?
Recall the identity: $(x + y)(x - y) = x^2 - y^2 \ \forall \ x, y \in \mathbb{C}$. We can write:

$$(a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2.$$

By multiplying a complex number by its conjugate, we can "eliminate" its imaginary part. Now, we can apply this to divisions with complex numbers.

Note 1.4 - Dividing Complex Numbers

Let $z = a + bi$ and $w = c + di$. Then, $z \div w = \frac{a+bi}{c+di}$. We can multiply the top and bottom by w's conjugate, $\overline{w} = c - di$. Then,

- $\frac{a+bi}{c+di} \cdot \frac{c-di}{c-di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(a+bi)(c-di)}{c^2+d^2}$.
- We already know how to evaluate the numerator.
- The denominator simply becomes a real number; this divides both the real and imaginary part of the numerator.

For example, $\frac{1+i}{4-i} = \frac{1+i}{4-i} \cdot \frac{4+i}{4+i} = \frac{(1+i)(4+i)}{(4-i)(4+i)} = \frac{4+i+4i-1}{16+1} = \frac{3+5i}{17} = \frac{3}{17} + \frac{5}{17}i$.

Example 1.2 - Operating on Complex Numbers

Suppose $z = 3 - 6i, w = 5 + i$.
(a) What is $\overline{z} - w$?
(b) What is $z \cdot w$?
(c) What is $\frac{z-i}{w+i}$?

Solution to (a): The conjugate of z is $\overline{z} = 3 + 6i$.
$\overline{z} - w = (3 + 6i) - (5 + i) = (3 - 5) + (6 - 1)i = \boxed{-2 + 5i}$.

Solution to (b): $z \cdot w = (3-6i)(5+i) = 3 \cdot 5 + 3 \cdot i + (-6i) \cdot 5 + (-6i) \cdot i = 15 + 3i - 30i - 6i^2 = 15 - 27i + 6 = \boxed{21 - 27i}$.

Solution to (c): $z - i = 3 - 6i - i = 3 - 7i$, $w + i = 5 + i + i = 5 + 2i$.
$\frac{z-i}{w+i} = \frac{3-7i}{5+2i}$. We multiply numerator and denominator by the conjugate of the denominator, $5 - 2i$:
$\frac{3-7i}{5+2i} \cdot \frac{5-2i}{5-2i} = \frac{(3-7i)(5-2i)}{(5+2i)(5-2i)}$
Numerator: $3 \cdot 5 + 3 \cdot (-2i) + (-7i) \cdot 5 + (-7i) \cdot (-2i) = 15 - 6i - 35i + 14i^2 = 15 - 41i - 14 = 1 - 41i$
Denominator: $(5)^2 - (2i)^2 = 25 - 4i^2 = 25 - (-4) = 29$
Therefore, $\frac{1-41i}{29} = \boxed{\frac{1}{29} - \frac{41}{29}i}$.

Now that we understand what complex numbers are and how to perform operations with them, we must somehow visually represent them. For the set of real numbers, we can easily represent them on a number line.

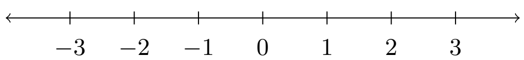


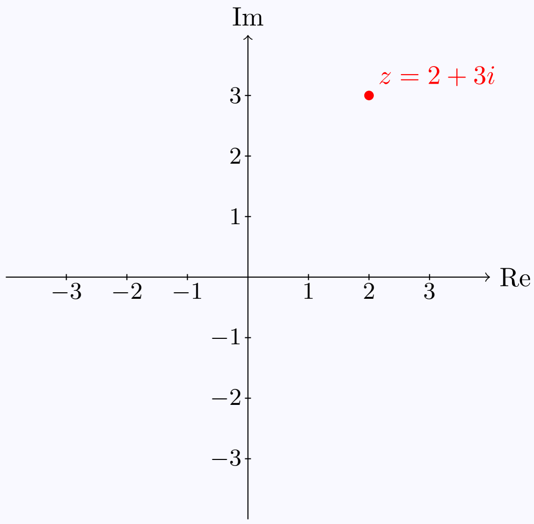
Figure 1.3: A Number Line for $\mathbb{R}$

A simple number line contains every single element of $\mathbb{R}$, but how would we represent a complex number? One way to think about complex numbers is to treat the real and imaginary parts separately. Suppose we let the real part be the x-coordinate and imaginary part be the y-coordinate. Then, we might be able to graph a complex number on a plane with two axes. That plane is called the **complex plane.**

Note 1.5 - The Complex Plane

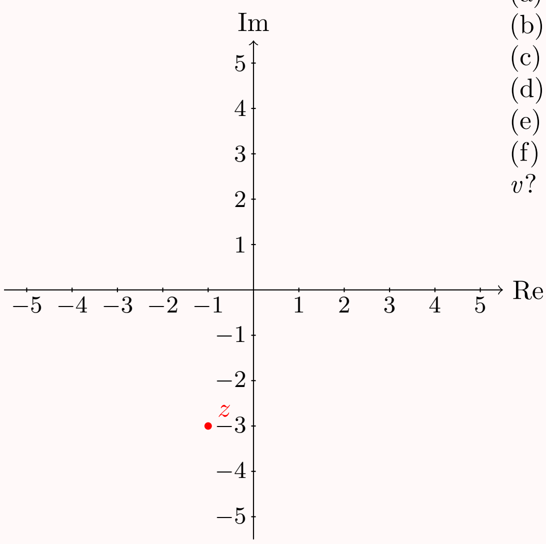
We represent complex numbers in the **complex plane**, which resembles the Cartesian Plane. When plotting a complex number $z = a + bi$,

- The horizontal axis is called the **Real Axis** (Denoted by Re)
- The vertical axis is called the **Imaginary Axis** (Denoted by Im)
- The horizontal coordinate corresponds to $\mathrm{Re}(z) = a$
- The vertical coordinate corresponds to $\mathrm{Im}(z) = b$



The real/imaginary part of z can be 0 as well (i.e. z can be a or bi where $a, b \in \mathbb{R}$). Hence, you can also plot any real or purely imaginary number on the complex plane.

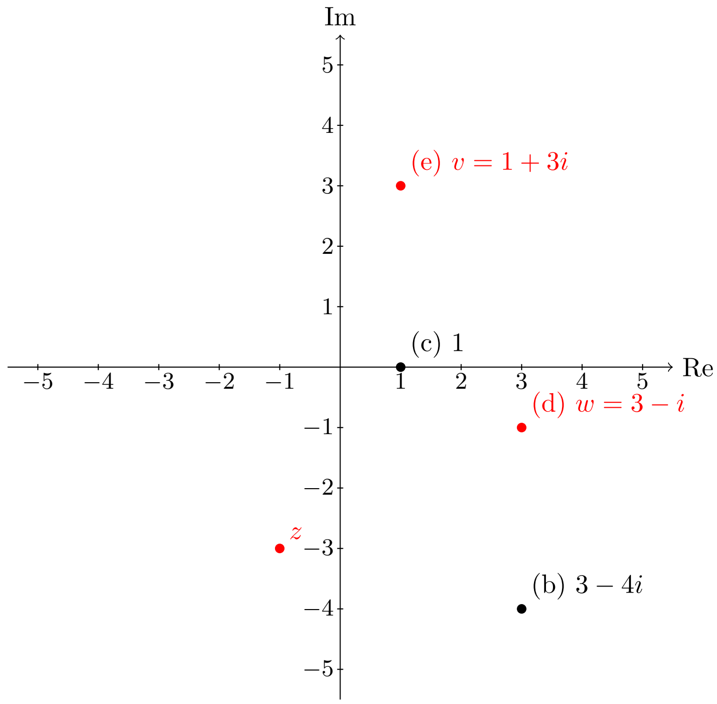
Example 1.3 - Plotting Complex Numbers



(a) What is z?
(b) Plot $3 - 4i$ on the plane.
(c) Plot 1 on the plane.
(d) Let $w = z \cdot i$. Plot w on the plane.
(e) Let $v = w \cdot i$. Plot v on the plane.
(f) What do you notice about z, w, and v?

Solution to (a): Since z is plotted at $(-1, -3)$, $z = \boxed{-1-3i}$.

Solutions to (b),(c),(d),(e):

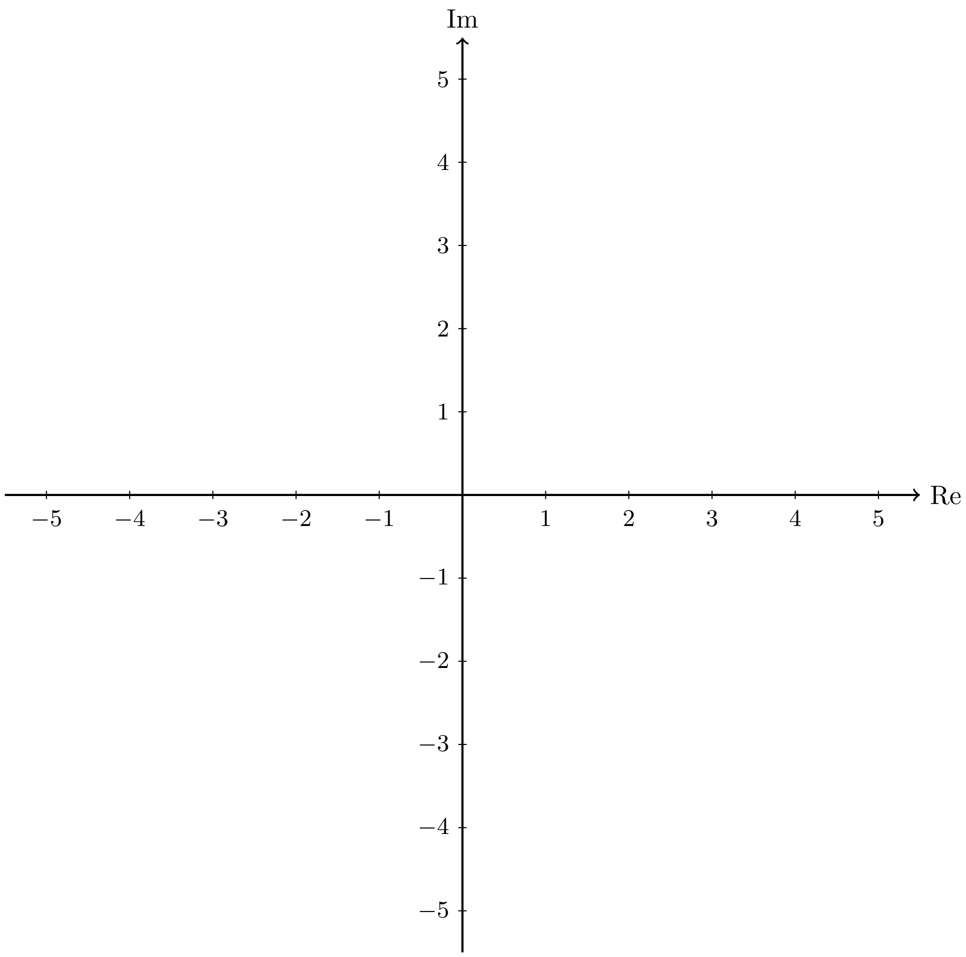


Solution to (f): One might notice that w is a 90-degree rotation of z, and v is a 90-degree rotation of w. This holds for all complex numbers. Multiplying a complex number by i is the same as rotating it counterclockwise by 90 degrees about the origin.

Section 1 Problems

For problems 1 - 5, evaluate the following if $z = 2 + i$ and $w = 1 - 2i$. Then, plot the resulting complex numbers on the complex plane at the bottom (label the points with the problem number):

1. $z + w$
2. $z - 2w$
3. $z \cdot w$
4. wi
5. $z \div \overline{w}$



Solutions to Section 1 Problems

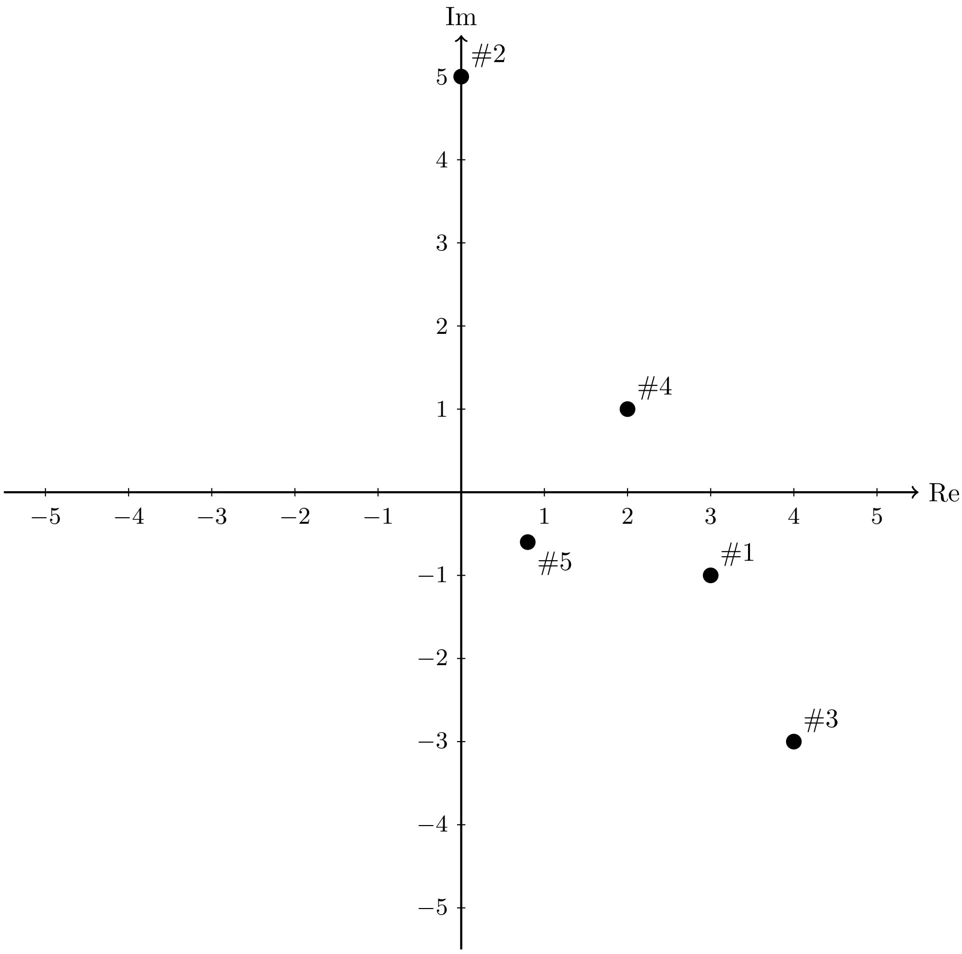
1. $z + w = (2+i) + (1-2i) = (2+1) + (1-2)i = \boxed{3-i}$

2. $z - 2w = (2+i) - 2(1-2i) = 2 + i - 2 + 4i = (2-2) + (1+4)i = \boxed{5i}$

3. $z \cdot w = (2+i)(1-2i) = 2\cdot 1 + 2\cdot(-2i) + i\cdot 1 + i\cdot(-2i) = 2 - 4i + i - 2i^2 = 2 - 4i + i + 2 = \boxed{4-3i}$

4. $wi = (1-2i)i = i - 2i^2 = i + 2 = \boxed{2+i}$

5. $z \div \overline{w}$: $\overline{w} = 1 + 2i$ so $\frac{2+i}{1+2i}$. Multiply num/den by $1-2i$ (which is the conjugate of $\overline{w}$):
   $\frac{2+i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{(2+i)(1-2i)}{(1+2i)(1-2i)}$
   Numerator: $2\cdot 1 + 2\cdot(-2i) + i\cdot 1 + i\cdot(-2i) = 2 - 4i + i - 2i^2 = 2 - 4i + i + 2 = 4 - 3i$
   Denominator: $1^2 - (2i)^2 = 1 - 4i^2 = 1 + 4 = 5$
   So $\frac{2+i}{1+2i} = \frac{4-3i}{5} = \boxed{\frac{4}{5} - \frac{3}{5}i}$



2 Characteristics of Complex Numbers

A complex number can also be described with its other components besides the real and imaginary parts. Let us first discuss the **modulus**.

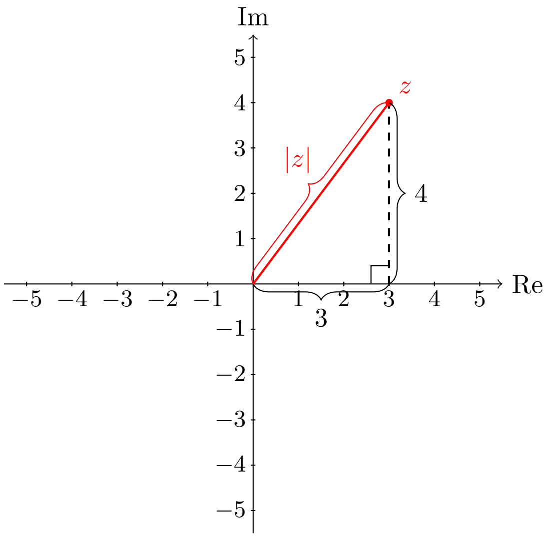
Note 2.1 - The Modulus of a Complex Number

The modulus of a complex number z is denoted by $|z|$, where $|z|$ indicates the distance from the origin to z in the complex plane. If $z = a + bi$, the formula is

$$|z| = \sqrt{a^2 + b^2}$$

The modulus is also called a **norm**, **magnitude**, or **absolute value** of a complex number.

We can show this formula by using the **Pythagorean Theorem**. Consider a complex plane where $z = a + bi$ is plotted:



Here, we can see that $|z|$ is simply the length of the hypotenuse of a right triangle with legs a and b. According to the Pythagorean Theorem, $a^2 + b^2 = |z|^2$. Applying the square root on both sides gives $|z| = \sqrt{a^2 + b^2}$. This formula works for any value of a and b, even if they are negative, since the distance must always be positive. In this case, we can see that $z = 3 + 4i$; therefore, $|z| = \sqrt{3^2 + 4^2} = 5$.

There are a few properties of the modulus that hold for all $z, w \in \mathbb{C}$:

1. $|z| = |-z|$
2. $|z \cdot w| = |z| \cdot |w|$
3. $|z \div w| = \frac{|z|}{|w|}$
4. $|z| = |\overline{z}|$ if $\overline{z}$ is the conjugate of z
5. $|z^n| = |z|^n$ if $n \in \mathbb{Z}$

We will take a deeper look at #5 later in this packet.

Example 2.1 - The Complex Modulus

(a) Evaluate the modulus of $1-8i$.
(b) Show (prove) properties #1 and #2 in the previous page.
(c) Find the modulus of z^3 if $z=-3-i$.

Solution to (a): $|1-8i| = \sqrt{1^2+(-8)^2} = \boxed{\sqrt{65}}$

Solution to (b): Let $z=a+bi$, $w=c+di$

1. $-z=-a-bi$. Then, $|-z| = \sqrt{(-a)^2+(-b)^2} = \sqrt{a^2+b^2}$, which is equivalent to $|z|$.
2. $z\cdot w = (a+bi)(c+di) = ac-bd+(ad+bc)i$. $|z\cdot w| = \sqrt{(ac-bd)^2+(ad+bc)^2} = \sqrt{(ac)^2-2abcd+(bd)^2+(ad)^2+2abcd+(bc)^2} = \sqrt{(ac)^2+(bd)^2+(ad)^2+(bc)^2}$. We can see that $|z|\cdot|w| = \sqrt{a^2+b^2}\cdot\sqrt{c^2+d^2} = \sqrt{(a^2+b^2)(c^2+d^2)} = \sqrt{(ac)^2+(ad)^2+(bc)^2+(bd)^2}$, so these two are equivalent.

Solution to (c): Using property #5, $|z^3| = |z|^3 = \left[\sqrt{(-3)^2+(-1)^2}\right]^3 = \boxed{10\sqrt{10}}$.

The modulus is crucial in representing a complex number in its polar form. A complex number's **argument** is also needed for this. We will take a look at what the polar form is later in this packet.

Note 2.2 - The Argument of a Complex Number

The argument of a complex number z is, by convention, denoted by $\arg(z)$. Geometrically, this represents the angle from the positive real axis to the vector representing z in radians. The function $\arg(z)$ is a **multi-valued function** and is called the **general argument**, since a full rotation would result in a coterminal angle that refers to the same complex number (this will be elaborated on in pages 12 - 13). Therefore, we use the single-valued function $\operatorname{Arg}(z)$. This is called the **principal argument** of z.

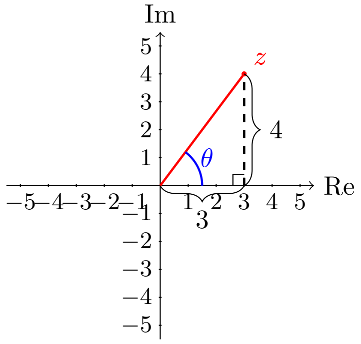
$$\arg(z) = \{\operatorname{Arg}(z) + 2n\pi \mid n \in \mathbb{Z}\}$$

If $z=a+bi$, the formula for $\operatorname{Arg}(z)$ is

$$\operatorname{Arg}(z) = \begin{cases} \text{undefined} & z=0 \\ \frac{\pi}{2} & a=0, b>0 \\ -\frac{\pi}{2} & a=0, b<0 \\ \tan^{-1}\left(\frac{b}{a}\right) & a>0 \\ \tan^{-1}\left(\frac{b}{a}\right)+\pi & a<0, b\geq 0 \\ \tan^{-1}\left(\frac{b}{a}\right)-\pi & a<0, b<0 \end{cases}$$

By convention, $\operatorname{Arg}(z) \in (-\pi, \pi]$

Do not be overwhelmed by the long formula above, the argument can be very easily calculated using intuition and careful thinking. You might notice that we use the inverse tangent function to calculate the argument. Let's figure out why this is by revisiting the figure we saw earlier:



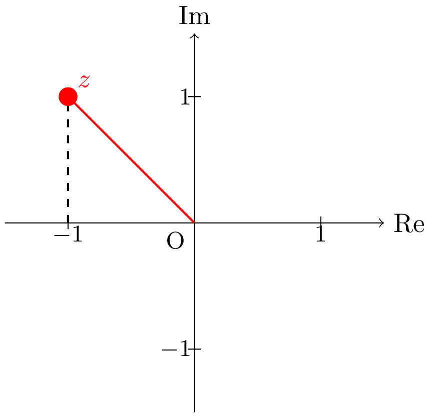
In the figure above, θ is the argument of z. Notice that since we already know $z = 3 + 4i$, we know the two legs of the right triangle. We can see that $\tan\theta = \frac{4}{3}$. To evaluate θ, we use the inverse:

$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$

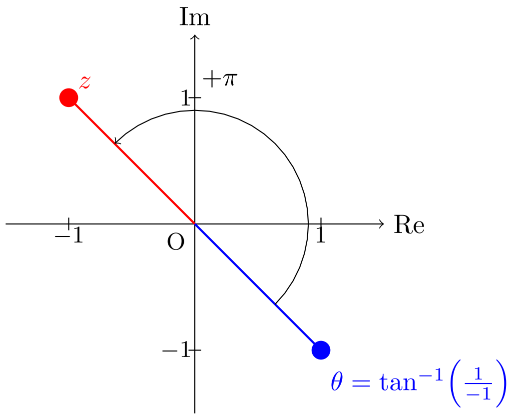
However, simply defining $\arg(z)$ as $\tan^{-1}\left(\frac{b}{a}\right)$ is not always correct. The reason is that

- the tangent function is periodic with period π, so $\tan\theta = \tan(\theta + \pi)$ and the value of $\tan\theta$ does not determine a unique angle;
- the expression $\frac{b}{a}$ only gives the ratio of b and a, but does not tell us in which quadrant the point (a, b) lies.

The function $\arctan\left(\frac{b}{a}\right)$ returns values only in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, corresponding to Quadrants I and IV. For complex numbers in Quadrants II and III (where $a < 0$), we must add or subtract π to move the angle into the correct quadrant. When $a = 0$, the ratio $\frac{b}{a}$ is undefined, so we must directly assign $\pm\frac{\pi}{2}$ (or say the argument is undefined when $z = 0$). This is why a piecewise definition of $\mathrm{Arg}(z)$ is necessary.



For example, suppose we have $z = -1 + i$, as shown above. Here, the formula $\arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$ gives us $\tan^{-1}\left(\frac{1}{-1}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$. But simply by looking at where z is located, we can tell that this is not the correct answer, as a complex number with angle $-\frac{\pi}{4}$ is located in Quadrant IV. To tackle this, we must add π (subtracting by π would also work, as what we need is an 180° rotation in either direction. However, we prefer to keep the principal value within the range of $(-\pi, \pi]$, so we add π instead).



So $\mathrm{Arg}(z) = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$. How about $\arg(z)$?. As we defined it previously, recall that

$$\arg(z) = \{\mathrm{Arg}(z) + 2n\pi \mid n \in \mathbb{Z}\}$$

Why do we add $2n\pi$, though? This is due to the fact that adding multiples of 360° would result in the same angle in standard position, as it is equivalent to a full rotation. To represent this in radians, we write $2n\pi, n \in \mathbb{Z}$. The figure below demonstrates this:

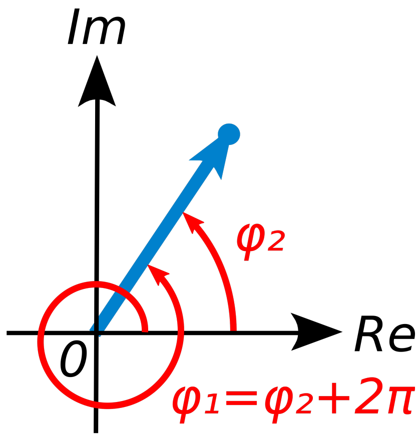
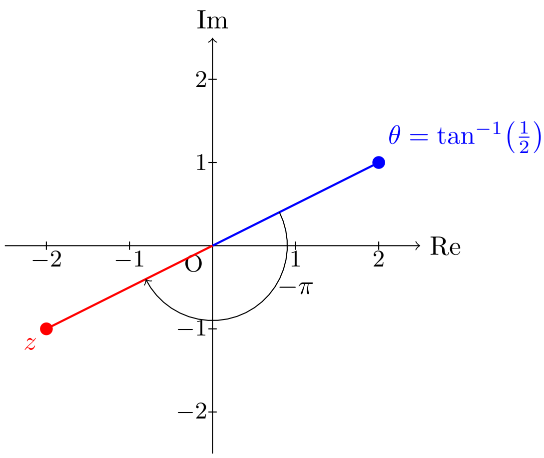


Figure 2.1: Fully Rotating a Complex Number

In the figure above, ψ_2 represents the principal argument of the complex number. Notice that the angle ψ_1, which is the result of adding 2π to ψ_2, refers to the same complex number. This can be generalized to the formula previously mentioned, as any multiple of 2π can be added.

Hence, $\arg(-1+i) = \boxed{\frac{3\pi}{4} + 2n\pi, n \in \mathbb{Z}}$.



What if the complex number is in Quadrant III? Take $-2-i$ as an example. $\tan^{-1}\left(\frac{-1}{-2}\right) = \tan^{-1}\left(\frac{1}{2}\right) \approx 0.464$ radians. However, this angle is in Quadrant I, while $-2-i$ is clearly located in Quadrant III. To move the angle to the correct quadrant, we must subtract π:

$$\operatorname{Arg}(-2-i) = \tan^{-1}\left(\frac{-1}{-2}\right) - \pi = \tan^{-1}\left(\frac{1}{2}\right) - \pi \approx -2.678 \text{ radians}$$

Therefore, $\arg(-2-i) = \boxed{\tan^{-1}\left(\frac{1}{2}\right) - \pi + 2n\pi, n \in \mathbb{Z}}$ or equivalently $\boxed{-2.678 + 2n\pi, n \in \mathbb{Z}}$.

For this section, note that many textbooks/sources use different notations, and $\arg(z)$ *might refer to the principal value and sometimes you might not have to include the "*$+2n\pi$*". However, in this packet, we aim to keep things as formal and conventional as possible.* There are a few properties of the argument that hold for all $z, w \in \mathbb{C}$:

1. $\arg(\overline{z}) = -\arg(z)$
2. $\arg(zw) = \arg(z) + \arg(w)$
3. $\arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w)$
4. $\arg(z^n) = n \cdot \arg(z), n \in \mathbb{Z}$

We will take a deeper look at #4 later in this packet.

Example 2.2 - The Argument

(a) Let $z = 4 - i$. Find the principal argument of z. Then find $\arg(z)$.
(b) Find $\arg(-2+6i)$.
(c) Suppose w is a complex number with modulus of 4 and argument $\frac{\pi}{2}$. Find w.
(d) Find $\arg(w_1 \div w_2)$ if $w_1 = 1 + i$ and $w_2 = -10i$.

Solution to (a): Since z is in Quadrant IV, we can simply use the formula $\tan^{-1}\left(\frac{-1}{4}\right) = \tan^{-1}(-4) \approx -1.326$ rad. Hence, $\arg(z) = \boxed{-1.326 + 2n\pi, n \in \mathbb{Z}}$.

Solution to (b): We can see that $-2+6i$ is located in Quadrant II, and $\tan^{-1}\left(\frac{6}{-2}\right) = \tan^{-1}(-3) \approx -1.249$. This is a smaller angle than $-\frac{\pi}{2}$, so it is located in Quadrant IV. Therefore, we add π. Hence, $\arg(-2+6i) = \boxed{-1.249 + \pi + 2n\pi, n \in \mathbb{Z}}$.

Solution to (c): If a complex number has an argument of $\frac{\pi}{2}$, then it is located on the positive imaginary axis. If w has a modulus of 4, it would be $\boxed{4i}$. The polar form is used to directly calculate the corresponding complex number for any modulus/argument, which we will cover in the next chapter.

Solution to (d): Use property #3: $\arg(w_1 \div w_2) = \arg(w_1) - \arg(w_2) = \arg(1+i) - \arg(-10i) = \tan^{-1}(1) - \left(-\frac{\pi}{2}\right) = \frac{\pi}{4} + \frac{\pi}{2} = \boxed{\frac{3\pi}{4}}$

Section 2 Problems

For problems 1 - 6, calculate the following:

1. $|3 - i|$
2. $\arg(4 + 7i)$
3. $\arg(3 - i)$
4. $\operatorname{Arg}(-i)$
5. $|-7 - 24i|$
6. $\operatorname{Arg}(-1 + 5i)$

Note that from here, many of the problems require clever thinking and are quite challenging, so it is completely fine if you aren't able to solve them. If you understood the concepts, you are expected to solve the problems above, however. Try to understand the solutions for all of the problems you missed.

7. Show that $\forall z \in \mathbb{C}, z \cdot \overline{z} = |z|^2$.

8. Let S denote the set of all complex numbers with a principal argument of π. Find the probability that a randomly chosen element of S has a nonzero imaginary part.

9. Let $z = -3 - 4i$ and suppose $\overline{z}w = -14 + 2i$. Find $|w|$ and $\arg(\overline{w})$.

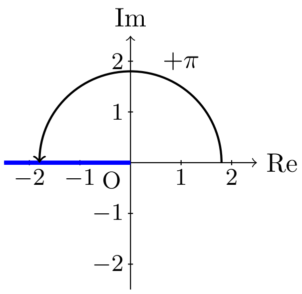
10. Suppose $|z_1 \cdot z_2| = 6$. If z_1 and z_2 both have integer moduli (plural of modulus). Find all possible pairs of $(|z_1|, |z_2|)$.

11. Let $w_1 = -1 - i$. It is given that $\arg(w_1) = -\arg(w_2)$ and $|w_2| = \sqrt{2}$. Suppose z is a complex number that resulted from the reflection of w_2 over the imaginary axis. Find $\arg(z^9)$.

12. **Challenge Problem:** (2004 AMC 12B #16) A function f is defined by $f(z) = i\overline{z}$, where $i = \sqrt{-1}$ and $\overline{z}$ is the complex conjugate of z. How many values of z satisfy both $|z| = 5$ and $f(z) = z$?

Solutions to Section 2 Problems

1. $|3-i| = \sqrt{3^2+1^2} = \boxed{\sqrt{10}}$
2. Since $4+7i$ is located in Quadrant I, we don't have to apply any manipulations. $\therefore \boxed{\tan^{-1}\left(\frac{7}{4}\right) + 2n\pi, n \in \mathbb{Z}}$.
3. $\arg(3-i)$ is located in Quadrant IV, so we don't have to apply any manipulations either. $\therefore \boxed{\tan^{-1}\left(\frac{1}{3}\right) + 2n\pi, n \in \mathbb{Z}}$.
4. By plotting $-i$ in the complex plane, we can easily see that the argument is $\boxed{-\frac{\pi}{2}}$. $\frac{3\pi}{2}$ is also acceptable, but recall that we prefer to keep the principal argument in the range of $(-\pi, \pi]$
5. $|-7-24i| = \sqrt{7^2+24^2} = \sqrt{625} = \boxed{25}$.
6. Notice that since $-1+5i$ is in Quadrant II, we have to add π. $\operatorname{Arg}(-1+5i) = \tan^{-1}\left(\frac{5}{-1}\right) + \pi = \boxed{\tan^{-1}(-5) + \pi}$.
7. Let $z = a+bi$. Then, $z \cdot \overline{z} = (a+bi)(a-bi) = a^2-(bi)^2 = a^2+b^2$. $|z| = \sqrt{a^2+b^2}$, so $|z|^2 = a^2+b^2$ as well.
8. S is a set of all complex numbers with a principal argument of π. Notice that if we plot this on the complex plane, we get a line:



Here, all the points on the line are negative real numbers, so they have an imaginary part of 0. Therefore, they cannot have a nonzero imaginary part, and the probability of a randomly chosen element of S having a nonzero imaginary part is $\boxed{0}$.

9. Let's try to find w. $\overline{z} = -3+4i$, so $w = \frac{-14+2i}{-3+4i} = \frac{-14+2i}{-3+4i} \cdot \frac{-3-4i}{-3-4i} = \frac{(-14+2i)(-3-4i)}{25} = \frac{50+50i}{25} = 2+2i$. We can see that $|w| = \sqrt{2^2+2^2} = \sqrt{8} = 2\sqrt{2}$. $\overline{w} = 2-2i$, which is located in Quadrant IV, so we can simply use the formula $\arg(2-2i) = \tan^{-1}\left(\frac{-2}{2}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$. The correct answer is $\boxed{-\frac{\pi}{4} + 2n\pi}$.
10. We are given that $|z_1 \cdot z_2| = 6$. Using one of the properties of the modulus, $|z_1| \cdot |z_2| = 6$. We know that $|z_1|$ and $|z_2|$ are integers (nonnegative, since the modulus can't be negative). Therefore, we just find pairs of factors of 6, which are $\boxed{(1,6), (2,3), (3,2), (6,1)}$.
11. The equation $\arg(w_1) = -\arg(w_2)$ tells us that they could be a conjugate pair if they have the same modulus. We are also given that $|w_2| = \sqrt{2}$, which is equal to $|w_1| = |-1-i| = \sqrt{1^2+1^2} = \sqrt{2}$. Therefore, w_1 and w_2 are conjugate pairs; $w_2 = -1+i$. z is a reflection of w_2 over the imaginary axis, so $z = 1+i$. We use one of the properties of the argument to evaluate $\arg(z^9)$ as $9 \cdot \arg(z) = 9 \cdot \tan^{-1}\left(\frac{1}{1}\right) = \frac{9\pi}{4}$. The general solution is $\boxed{\frac{9\pi}{4} + 2n\pi, n \in \mathbb{Z}}$.

Before we move on to the next problem, addressing the following theorem is necessary:

Theorem 2.1 - Equality of Complex Numbers

Two complex numbers A and B are equal if and only if $\text{Re}(A) = \text{Re}(B)$ and $\text{Im}(A) = \text{Im}(B)$.

We will use this theorem to solve #12.

12. Let $z = a + bi$. Then, $f(z) = i(a - bi) = ai - bi^2 = b + ai$. If $f(z) = z$, then $a + bi = b + ai$. From Theorem 2.1, we know that $\text{Re}(a + bi) = \text{Re}(b + ai)$ and $\text{Im}(a + bi) = \text{Im}(b + ai)$. Therefore, $a = b$. If $|z| = 5$, then $\sqrt{a^2 + b^2} = 5$. Substitute a for b, and we get $\sqrt{2b^2} = 5$. So $2b = \pm 5, b = \pm \frac{5}{2}$. Since $a = b$, possible pairs of (a, b) are $\left(\frac{5}{2}, \frac{5}{2}\right)$ and $\left(-\frac{5}{2}, -\frac{5}{2}\right)$. Therefore, there are $\boxed{2}$ possible values of z.

Theorem 2.1 can be used to solve many problems, including evaluating $\sqrt{i}$. We will take a look at this later.

3 The Polar Form and Euler's Formula

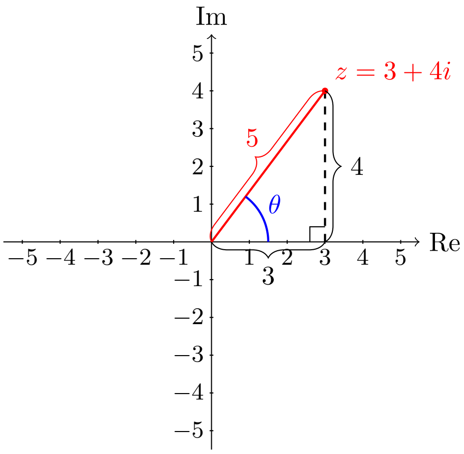
So far, we've been representing complex numbers in their **rectangular form** (also called **Cartesian form**): $z = a + bi$. However, there's another powerful way to represent complex numbers that uses distance and angle instead of x and y coordinates. This is called the **polar form**.

Note 3.1 - Understanding Polar Coordinates

In polar coordinates, we describe a point using:

- **r**: The distance from the origin (this is the modulus $|z|$)
- θ: The angle from the positive real axis (this is the argument $\arg(z)$)

This is the reason why we covered the modulus and the argument in the previous chapter. In polar coordinates, we denote a point as (r, θ)



In the above example, we can see that $r = \sqrt{3^2 + 4^2} = 5$, and $\theta = \arg(3+4i) = \tan^{-1}\left(\frac{4}{3}\right) \approx 0.927$. Using polar coordinates, we would represent $3 + 4i$ as $(5, 0.927)$. In most cases, we only put the principal argument when dealing with polar forms. However, when we are looking for the complex roots or the question asks for all possible polar representations, we find the general solution $\arg(z)$.

Note 3.2 - Converting Between Rectangular and Polar Form

For a complex number $z = a + bi$: **From rectangular to polar:**

- $r = |z| = \sqrt{a^2 + b^2}$ (distance from origin)
- $\theta = \arg(z)$ (angle from positive real axis)

From polar to rectangular:

- $a = r\cos(\theta)$ (horizontal coordinate)
- $b = r\sin(\theta)$ (vertical coordinate)

This gives us the polar form: $z = r(\cos\theta + i\sin\theta)$. If you distribute and evaluate everything, it will result in $a + bi$.

Why does this work? Look at the right triangle formed by z, the real axis, and the line from the origin to z. By definition:

- $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{r}$, so $a = r\cos(\theta)$
- $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{r}$, so $b = r\sin(\theta)$

Therefore: $a + bi = r\cos(\theta) + ir\sin(\theta) = r(\cos(\theta) + i\sin(\theta))$. We sometimes abbreviate this as $r\,\text{cis}(\theta)$.
We found out that $3 + 4i$ in its polar coordinates is $(5, 0.927)$. Let's confirm this. Using the formula $r\,\text{cis}\,\theta$, the rectangular form would be $5\cos(0.927) + i5\sin(0.927)$. Using a calculator, $\cos(0.927) \approx 0.6$, and $\sin(0.927) \approx 0.8$. Therefore, the expression evaluates to $5 \cdot 0.6 + 5i \cdot 0.8 = 3 + 4i$.

Example 3.1 - Converting to Polar Form

Convert the following complex numbers to polar form:
(a) $z = -3 + 4i$
(b) $z = -2 + 2i$
(c) $z = -5i$
(d) $z = 7$

Solution to (a): For $z = -3 + 4i$:

- $r = \sqrt{(-3)^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$
- Since z is in Quadrant II: $\theta = \tan^{-1}\left(\frac{4}{-3}\right) + \pi \approx 2.215$.

So $z = \boxed{5(\cos -2.215 + i\sin -2.215)}$

Solution to (b): For $z = -2 + 2i$:

- $r = \sqrt{(-2)^2 + 2^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$
- Since z is in Quadrant II: $\theta = \tan^{-1}\left(\frac{2}{-2}\right) + \pi = \tan^{-1}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

So $z = \boxed{2\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})}$

Solution to (c): For $z = -5i$:

- $r = \sqrt{0^2 + (-5)^2} = 5$
- $\theta = -\frac{\pi}{2}$ (pointing straight down)

So $z = \boxed{5(\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2}))}$

Solution to (d): For $z = 7$:

- $r = \sqrt{7^2 + 0^2} = 7$
- $\theta = 0$ (pointing right along positive real axis)

So $z = \boxed{7(\cos 0 + i\sin 0)}$

Example 3.2 - Converting from Polar to Rectangular Form

Convert the following from polar to rectangular form:
(a) $z = 6(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$
(b) $z = 2\sqrt{2}(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})$

Solution to (a): $z = 6(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

- $\cos\frac{\pi}{3} = \frac{1}{2}$ and $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$
- $a = 6 \cdot \frac{1}{2} = 3$
- $b = 6 \cdot \frac{\sqrt{3}}{2} = 3\sqrt{3}$

So $z = \boxed{3 + 3\sqrt{3}i}$.

Solution to (b): $z = 2\sqrt{2}(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4})$

- $\cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$ and $\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$
- $a = 2\sqrt{2} \cdot \left(-\frac{\sqrt{2}}{2}\right) = -1$
- $b = 2\sqrt{2} \cdot \left(-\frac{\sqrt{2}}{2}\right) = -1$

So $z = \boxed{-1 - i}$.

Note 3.3 - Operations in Polar Form

Polar form makes certain operations much easier:
Multiplication: If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then:

$$z_1 \cdot z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2)$$

Division: If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)) = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

Why do these formulas work?
For multiplication:

$$z_1 \cdot z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2[\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2]$$

$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)]$$

$$= r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

The last step uses the trigonometric identities:

- $\cos(A + B) = \cos A\cos B - \sin A\sin B$
- $\sin(A + B) = \sin A\cos B + \cos A\sin B$

For division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2}$$

$$= \frac{r_1}{r_2}\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} \cdot \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2}\frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}$$

Here, we use the Pythagorean Identity $\sin^2 x + \cos^2 x = 1$; so the denominator, $\cos^2\theta_2 + \sin^2\theta_2$, is 1.

$$= \frac{r_1}{r_2}(\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2)$$

$$= \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)]$$

$$= \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

The last step uses the trigonometric identities:

- $\cos(A - B) = \cos A\cos B + \sin A\sin B$

- $\sin(A - B) = \sin A \cos B - \cos A \sin B$

Example 3.3 - Operations in Polar Form

Let $z_1 = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ and $z_2 = 3(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$.
(a) Find $z_1 \cdot z_2$
(b) Find $\frac{z_1}{z_2}$

Solution to (a): Using the multiplication formula:

$$z_1 \cdot z_2 = 2 \cdot 3(\cos(\frac{\pi}{6} + \frac{\pi}{3}) + i\sin(\frac{\pi}{6} + \frac{\pi}{3}))$$

$$= 6(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 6(0 + i \cdot 1) = \boxed{6i}$$

Solution to (b): Using the division formula:

$$\frac{z_1}{z_2} = \frac{2}{3}(\cos(\frac{\pi}{6} - \frac{\pi}{3}) + i\sin(\frac{\pi}{6} - \frac{\pi}{3}))$$

$$= \frac{2}{3}(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})) = \frac{2}{3}(\frac{\sqrt{3}}{2} - i \cdot \frac{1}{2}) = \boxed{\frac{\sqrt{3}}{3} - \frac{1}{3}i}$$

Notice how much easier these calculations are compared to using FOIL.
Now, let's take a look at how the polar form simplifies exponentiation of complex numbers.

Theorem 3.1 - De Moivre's Theorem

The De Moivre's Theorem states that if $z = r(\cos\theta + i\sin\theta)$ and n is an integer, then:

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta)) = r^n \operatorname{cis}(n\theta)$$

Proving this theorem takes a lot of effort and complicated methods, such as mathematical induction. We will not cover the proof in this packet, but feel free to research more about it if you want.
De Moivre's Theorem for Roots and Roots of Unity are applications of the polar form, which we will cover in the next chapter. De Moivre's Theorem can be used to simplify another operation of complex numbers (exponentiation), as shown in the example below.

Example 3.4 - De Moivre's Theorem

(a) Find $(1 + i\sqrt{3})^6$
(b) Find $\left(\operatorname{cis}\frac{\pi}{6}\right)^{18}$

Solution to (a): We first convert $1 + i\sqrt{3}$ to the polar form.

- $r = \sqrt{1^2 + \sqrt{3}^2} = 2.$
- $\theta = \arctan\left(\frac{\sqrt{3}}{1}\right) = \arctan(\sqrt{3}) = \frac{\pi}{3}.$

Therefore, $1 + i\sqrt{3} = 2\operatorname{cis}\left(\frac{\pi}{3}\right)$.
Using De Moivre's Theorem, $(2\operatorname{cis}\frac{\pi}{3})^6 = 2^6 \operatorname{cis}\left(\frac{\pi}{3} \cdot 6\right) = 2^6 \operatorname{cis}(2\pi) = 64(\cos 2\pi + i\sin 2\pi) = 64(1 + 0) = 64$. Hence, the answer is $\boxed{64}$.

Solution to (b): $\left(\operatorname{cis}\frac{\pi}{6}\right)^{18} = 1^{18}\operatorname{cis}\left(\frac{\pi}{6} \cdot 18\right) = \operatorname{cis}(3\pi) = \cos 3\pi + i\sin 3\pi = -1$. Hence, the answer is $\boxed{-1}$.

Theorem 3.2 - Euler's Formula

One of the most beautiful and important formulas in mathematics is **Euler's Formula**:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This formula connects the exponential function with trigonometry. Using Euler's formula, we can write polar form even more compactly:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

This is called the **exponential form** of a complex number.

Proof of Euler's Formula: Consider the Taylor series expansions:

- $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

If these formulas seem very obscure and unfamiliar, that is fine. This concept is usually covered in Calculus II, but for now, know that these are alternate forms of the functions $e^x, \cos x$, and $\sin x$. These are infinite sums that utilize the factorial function $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$.
If we substitute $x = i\theta$ into the exponential series:

$$\begin{aligned} e^{i\theta} &= 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \end{aligned}$$

Notice that the form above matches the Taylor Series definitions of $\cos\theta$ and $\sin\theta$. Therefore, this is equivalent to

$$\cos\theta + i\sin\theta$$

If this all sounds so difficult and abstract, that is totally fine. For most cases, you just have to know this alternate representation of complex numbers:

$$z = r\,\mathrm{cis}\,\theta = re^{i\theta}$$

Euler's Identity: When $\theta = \pi$, we get the famous Euler's Identity:

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + i\cdot 0 = -1$$

Or: $e^{i\pi} + 1 = 0$
This is often called "the most beautiful equation in mathematics" because it connects five fundamental constants: e, i, π, 1, and 0.

Example 3.5 - Euler's Formula

(a) Represent $-1 + i$ in its exponential form.
(b) Convert $4e^{i\frac{\pi}{3}}$ to the rectangular form.

Solution to (a): $|-1+i| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$
$\arg(-1+i) = \tan^{-1}\left(\frac{1}{-1}\right) + \pi = \tan^{-1}(-1) + \pi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$.
Therefore, the exponential form is $\sqrt{2}e^{i\frac{3\pi}{4}}$. Here, a more accurate answer would be $\sqrt{2}e^{i\frac{3\pi}{4}+2n\pi}, n \in$

$\mathbb{Z}$, but this is the general solution and is not specifically asked for in the question.

Solution to (b): $4e^{i\frac{\pi}{3}} = 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 4(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 2 + 2i\sqrt{3}$.

Theorem 3.3 - Fundamental Theorem of Algebra

Before moving on to the next theorem, it is important to cover the **Fundamental Theorem of Algebra**. The Fundamental Theorem of Algebra states that for any polynomial

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

Where the coefficients a_i are complex numbers, $a_n \neq 0$, and $n \geq 1$, there exists at least one complex number r such that $P(r) = 0$. A direct consequence of this is that the polynomial can be factored completely into linear factors:

$$P(z) = a_n(z - r_1)(z - r_2)\cdots(z - r_{n-1})(z - r_n)$$

Therefore, its corollary states that a polynomial $P(z)$ of degree n has n complex roots.

The Fundamental Theorem of Algebra is quite important in discussing the next theorem.

Theorem 3.4 - De Moivre's Theorem for Roots

The De Moivre's Theorem for Roots is an extension of Theorem 3.1 (De Moivre's Theorem). It is used to find the nth root of a complex number. Typically, it is used to solve equations in the form of $x^n = z$ to find values of x that satisfy the equation for the complex number z. It states that if $z = r(\cos\theta + i\sin\theta)$, then:

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right) = \sqrt[n]{r}\,\mathrm{cis}\left(\frac{\theta + 2\pi k}{n}\right)$$

Here, n is an integer and k is an integer that goes from 0 to $n-1$ to find the n distinct roots.

Notice that there are exactly n complex solutions to the equation $x^n = z$. This means there are n zeros for the polynomial $x^n - z$ of degree n.

Example 3.6 - Finding Complex Roots

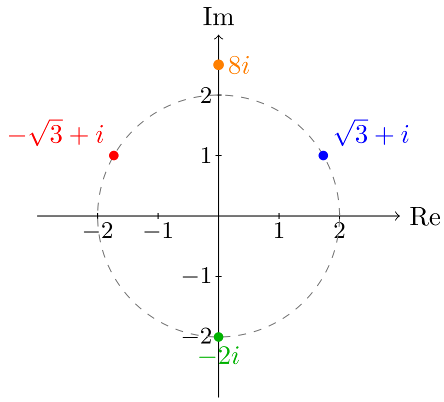
(a) Find all cube roots of $8i$.
(b) Find all fourth roots of $1 - i$.

Solution to (a): First, write $8i$ in polar form: $8i = 8(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$
The cube roots are:

$$\sqrt[3]{8i} = \sqrt[3]{8}\left(\cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)\right)$$
$$= 2\left(\cos\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right)$$

For $k = 0, 1, 2$:

- $k = 0$: $2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2(\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}) = \sqrt{3} + i$
- $k = 1$: $2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) = 2(-\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}) = -\sqrt{3} + i$
- $k = 2$: $2(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}) = 2(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = 2(0 - i) = -2i$



Notice that the three roots are equally spaced around a circle! This is always true for n-th roots of complex numbers. They lie on a circle with center origin and radius $r^{\frac{1}{n}}$.

Solution to (b): First, write $1 - i$ in polar form: $1 - 1i =$
The cube roots are:

$$\sqrt[3]{8i} = \sqrt[3]{8}\left(\cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)\right)$$

$$= 2\left(\cos\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right)\right)$$

For $k = 0, 1, 2$:

- $k = 0$: $2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = 2(\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}) = \sqrt{3} + i$
- $k = 1$: $2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}) = 2(-\frac{\sqrt{3}}{2} + i\cdot\frac{1}{2}) = -\sqrt{3} + i$
- $k = 2$: $2(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}) = 2(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = 2(0 - i) = -2i$

Section 3 Problems

For problems 1 - 8, convert between rectangular and polar forms as indicated:

1. Convert $2+2i$ to polar form
2. Convert $3(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$ to rectangular form
3. Convert $-5+12i$ to polar form
4. Convert $4(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})$ to rectangular form
5. Convert $-i$ to polar form
6. Convert $6e^{i\frac{\pi}{6}}$ to rectangular form
7. Convert $-8+8i\sqrt{3}$ to exponential form
8. Convert $\sqrt{2}e^{i\frac{3\pi}{4}}$ to rectangular form
9. Find $(1+i)^8$ using De Moivre's Theorem
10. Find all fourth roots of 16
11. Find all square roots of $-9i$
12. Evaluate $(2\operatorname{cis}\frac{\pi}{3})^3 \div (\operatorname{cis}\frac{\pi}{6})^2$

13. Find all cube roots of 27

14. Solve $z^5 = -32$ for all complex values of z

Solutions to Section 3 Problems

1. For $2+2i$: $r=\sqrt{2^2+2^2}=\sqrt{8}=2\sqrt{2}$, $\theta=\tan^{-1}(\frac{2}{2})=\frac{\pi}{4}$. Therefore, $2+2i=\boxed{2\sqrt{2}(\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4})}$

2. $3(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})=3(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})=\boxed{\dfrac{3\sqrt{2}}{2}+\dfrac{3\sqrt{2}}{2}i}$

3. For $-5+12i$: $r=\sqrt{(-5)^2+12^2}=\sqrt{25+144}=\sqrt{169}=13$, $\theta=\tan^{-1}(\frac{12}{-5})+\pi\approx 1.966$. Therefore, $-5+12i=\boxed{13(\cos 1.966+i\sin 1.966)}$

4. $4(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3})=4(-\frac{1}{2}+i\frac{\sqrt{3}}{2})=\boxed{-2+2i\sqrt{3}}$

5. For $-i$: $r=1$, $\theta=-\frac{\pi}{2}$. Therefore, $-i=\boxed{\cos(-\dfrac{\pi}{2})+i\sin(-\dfrac{\pi}{2})}$

6. $6e^{i\frac{\pi}{6}}=6(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})=6(\frac{\sqrt{3}}{2}+i\frac{1}{2})=\boxed{3\sqrt{3}+3i}$

7. For $-8+8i\sqrt{3}$: $r=\sqrt{(-8)^2+(8\sqrt{3})^2}=\sqrt{64+192}=\sqrt{256}=16$, $\theta=\tan^{-1}(\frac{8\sqrt{3}}{-8})+\pi=\tan^{-1}(-\sqrt{3})+\pi=-\frac{\pi}{3}+\pi=\frac{2\pi}{3}$. Therefore, $-8+8i\sqrt{3}=\boxed{16e^{i\frac{2\pi}{3}}}$

8. $\sqrt{2}e^{i\frac{3\pi}{4}}=\sqrt{2}(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})=\sqrt{2}(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})=\boxed{-1+i}$

9. First convert $1+i$ to polar: $r=\sqrt{1^2+1^2}=\sqrt{2}$, $\theta=\frac{\pi}{4}$. Using De Moivre's Theorem: $(\sqrt{2}\operatorname{cis}\frac{\pi}{4})^8=(\sqrt{2})^8\operatorname{cis}(8\cdot\frac{\pi}{4})=16\operatorname{cis}(2\pi)=16(\cos 2\pi+i\sin 2\pi)=\boxed{16}$

10. $16=16(\cos 0+i\sin 0)$. Fourth roots: $\sqrt[4]{16}\operatorname{cis}(\frac{0+2\pi k}{4})=2\operatorname{cis}(\frac{\pi k}{2})$ for $k=0,1,2,3$. This gives: $\boxed{2, 2i, -2, -2i}$

11. $-9i=9(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2})$. Square roots: $\sqrt{9}\operatorname{cis}(\frac{\frac{3\pi}{2}+2\pi k}{2})=3\operatorname{cis}(\frac{3\pi}{4}+\pi k)$ for $k=0,1$. This gives: $\boxed{\dfrac{-3\sqrt{2}}{2}+\dfrac{3\sqrt{2}}{2}i, \dfrac{3\sqrt{2}}{2}-\dfrac{3\sqrt{2}}{2}i}$

12. $(2\operatorname{cis}\frac{\pi}{3})^3=8\operatorname{cis}\pi=-8$, $(\operatorname{cis}\frac{\pi}{6})^2=\operatorname{cis}\frac{\pi}{3}$. Therefore: $\frac{-8}{\operatorname{cis}\frac{\pi}{3}}=-8\operatorname{cis}(-\frac{\pi}{3})=-8(\cos(-\frac{\pi}{3})+i\sin(-\frac{\pi}{3}))=-8(\frac{1}{2}-i\frac{\sqrt{3}}{2})=\boxed{-4+4i\sqrt{3}}$

13. $27=27(\cos 0+i\sin 0)$. Cube roots: $\sqrt[3]{27}\operatorname{cis}(\frac{0+2\pi k}{3})=3\operatorname{cis}(\frac{2\pi k}{3})$ for $k=0,1,2$. This gives: $\boxed{3, -\dfrac{3}{2}+\dfrac{3\sqrt{3}}{2}i, -\dfrac{3}{2}-\dfrac{3\sqrt{3}}{2}i}$

14. $-32=32(\cos\pi+i\sin\pi)$. Fifth roots: $\sqrt[5]{32}\operatorname{cis}(\frac{\pi+2\pi k}{5})=2\operatorname{cis}(\frac{\pi+2\pi k}{5})$ for $k=0,1,2,3,4$. This gives: $\boxed{2\operatorname{cis}\dfrac{\pi}{5}, 2\operatorname{cis}\dfrac{3\pi}{5}, 2\operatorname{cis}\pi, 2\operatorname{cis}\dfrac{7\pi}{5}, 2\operatorname{cis}\dfrac{9\pi}{5}}$